Mixing time for the Solid-on-Solid model

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joint work with
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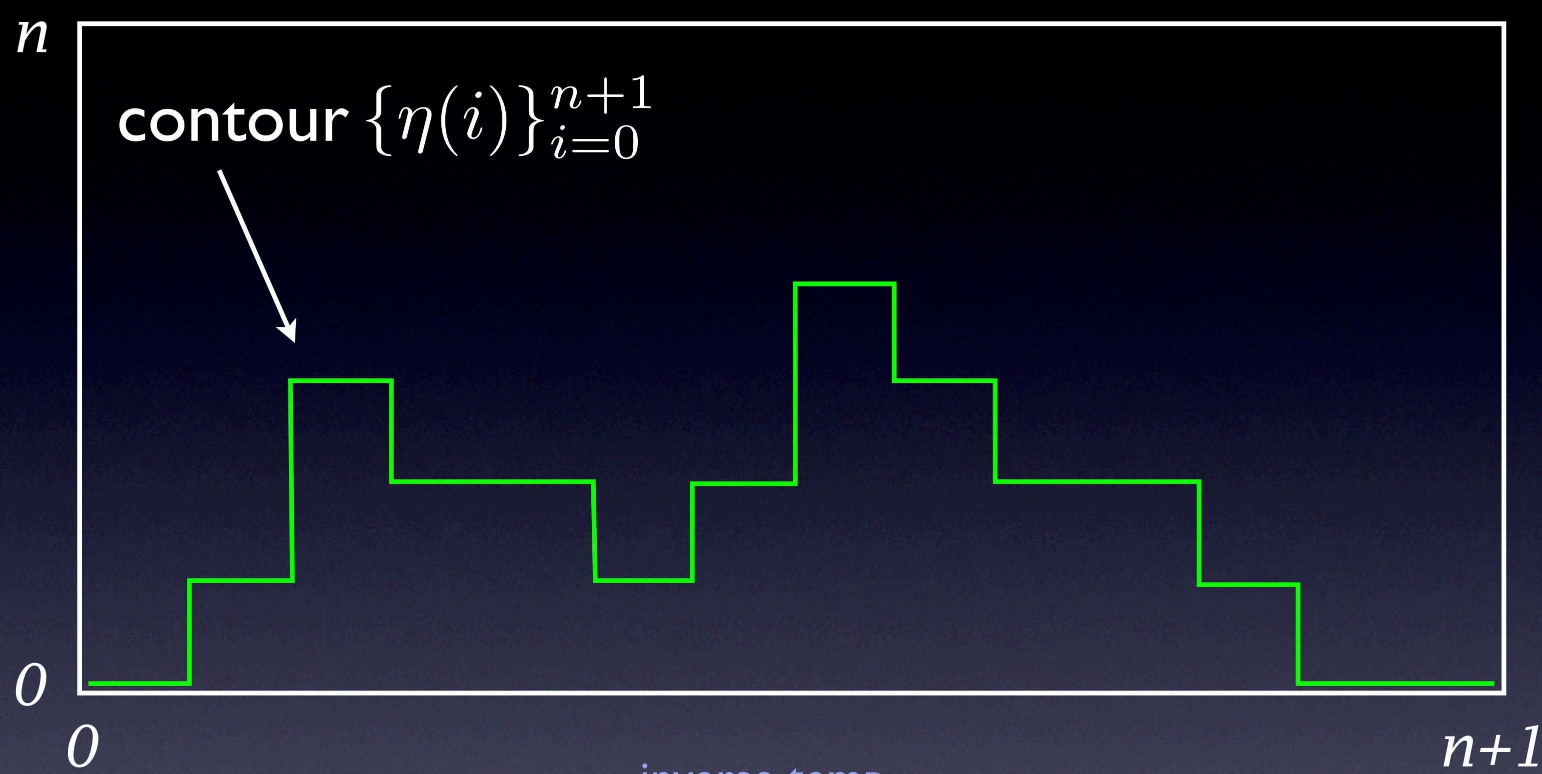
Solid-on-Solid (SOS) model

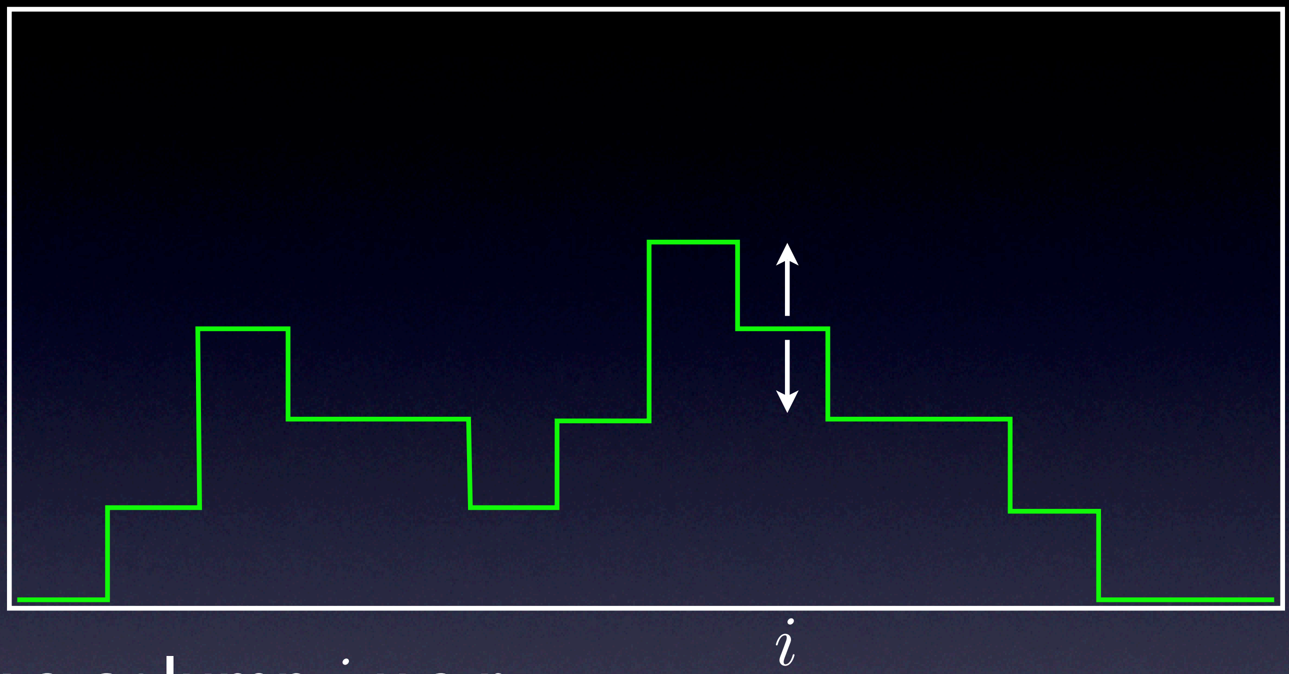


inverse temp.

Gibbs distribn: $\pi(\eta) = Z_\beta^{-1} \exp\left\{-\beta \sum_{i=1}^{n+1} |\eta(i) - \eta(i-1)|\right\}$

Boundary conditions: $\eta(0) = \eta(n+1) = 0$

Single-site (Glauber) dynamics



- pick a column i u.a.r.
- set $\eta(i) \in \{\eta(i), \eta(i) \pm 1\}$ with appropriate probs.
- $\eta_t \longrightarrow \pi$ as $t \longrightarrow \infty$

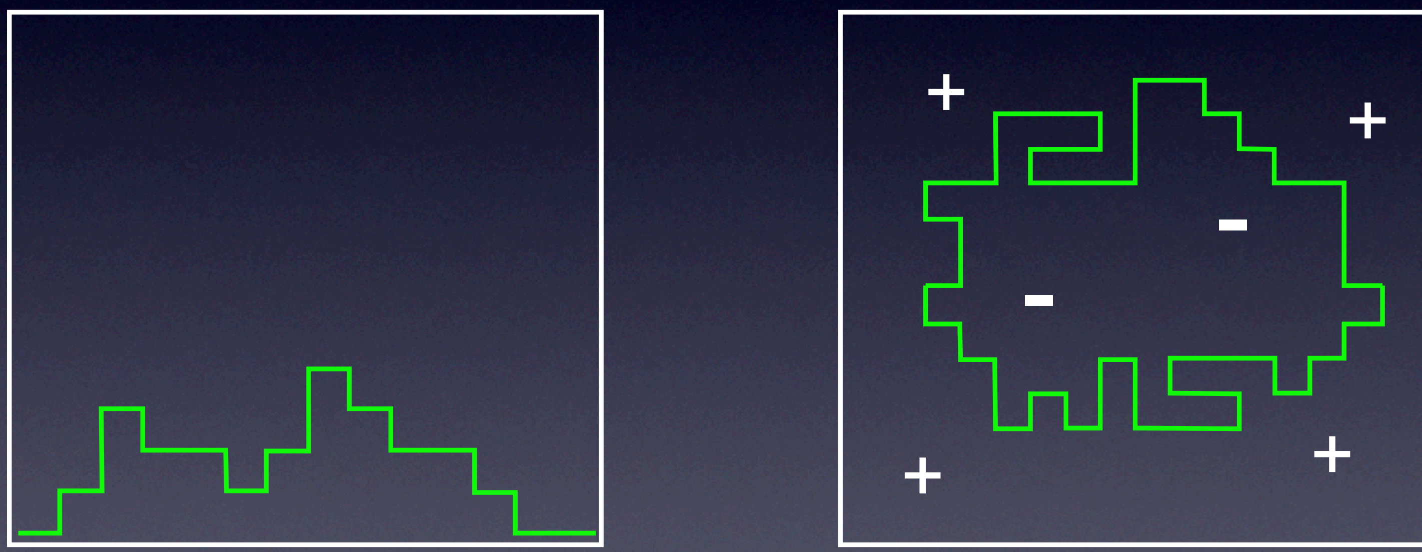
Goal: analyze the <u>mixing time</u>:

$$\tau = \min\{t : \|\eta_t - \pi\| \leq 1/4 \; \forall \eta_0\}$$

Why?

1. Model for random surfaces etc. [Privman/Švrakić...]

2. Connection with low temperature Ising model

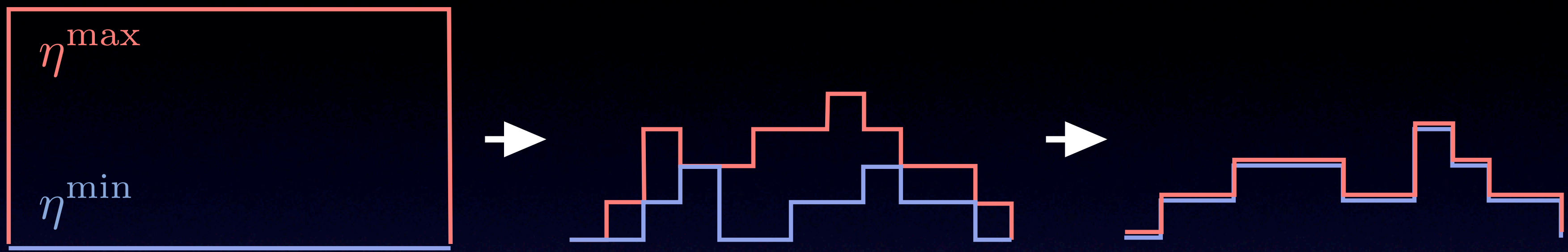


At low temps, few "overhangs" ⇒ good approximation

(Zero temp. solved by [Chayes/Schonmann/Swindle])

3. Challenge to existing techniques

- Monotone coupling [Propp/Wilson]



E[Δ area] ≤ 0 ⇒ mixing time = $\tilde{O}(n^5)$

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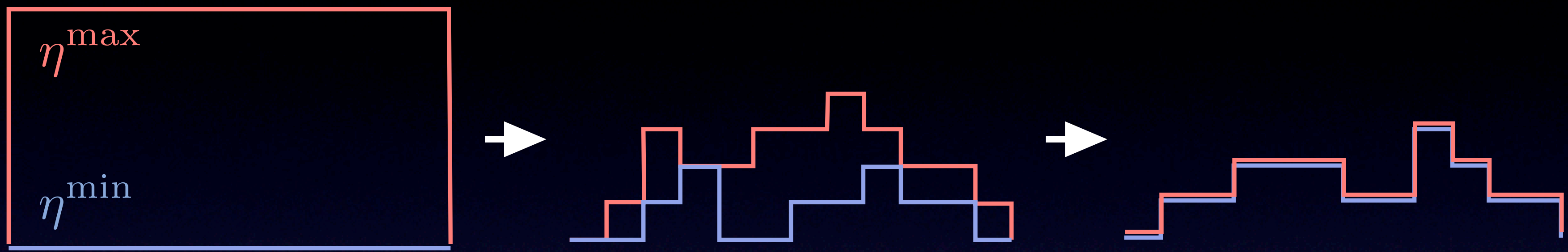

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- Comparison [Diaconis/Saloff-Coste]

  Compare with "non-local" dynamics:

  $\eta(i) \longrightarrow$ any value in $[0, n]$ w. prob. $\pi(\,\cdot \mid \eta(i \pm 1))$

  ⇒ Mixing time = $\tilde{O}(n^8)$

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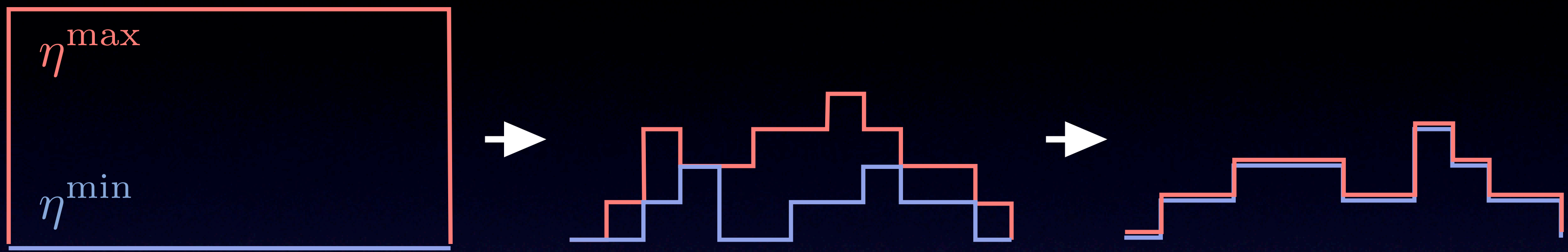

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⇒ Mixing time = $\tilde{O}(n^8)$

Both are loose and give little geometrical insight

Results

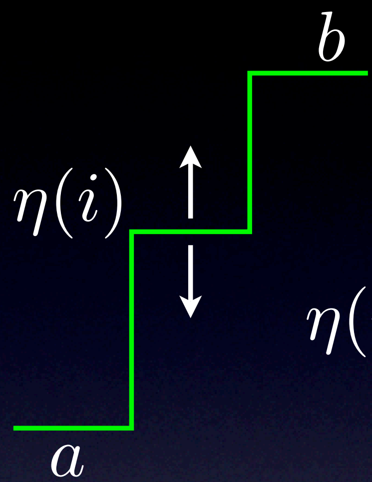
Main Theorem: For all $\beta > 0$, the single-site dynamics has mixing time $\tilde{O}(n^{3.5})$

Also: Almost matching lower bound $\Omega(n^3)$

Bonus: Analysis gives insight into actual evolution of contour

“It’s not a good idea to do a proof in a talk”
[D. Gamarnik, Phys. of Algorithms, 2009]

Non-local "Column" Dynamics



$$\eta(i) \to k \quad \text{w. prob.} \;\propto \begin{cases} \frac{1}{a-b} & a \le k \le b \\ e^{-\beta(a-k)} & 0 \le k < a \\ e^{-\beta(k-b)} & b < k \le n \end{cases}$$

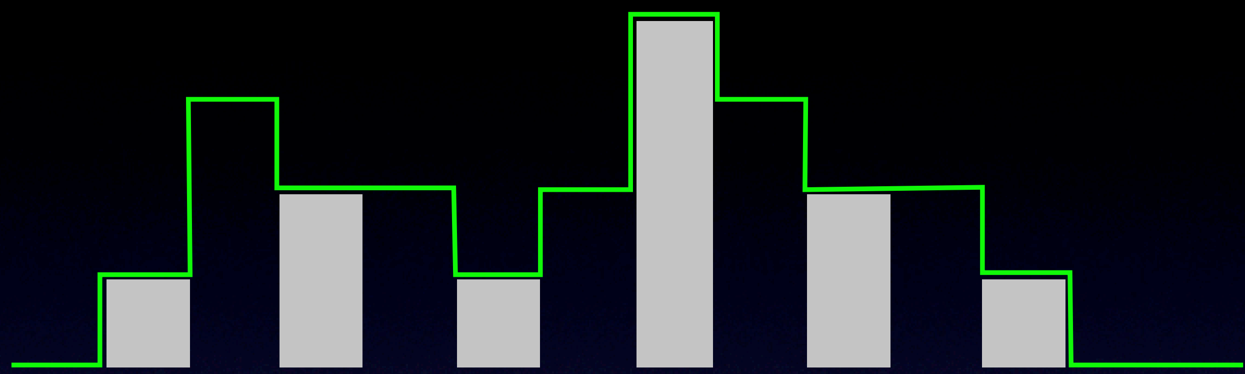
In absence of "walls" at height *0*, *n*, by symmetry:

$$\mathrm{E}[\eta(i)] = \tfrac{1}{2}[a+b] = \tfrac{1}{2}[\eta(i-1)+\eta(i+1)]$$

and mixing time $O(n^3 \log n)$ follows from 2nd eigenvalue of discrete Laplacian

<u>Can show</u> : repulsion from walls only helps!

Simulating Column Dynamics by Single-Site Dynamics



Mixing time of "odd-even" column dynamics is $\tilde{O}(n^2)$

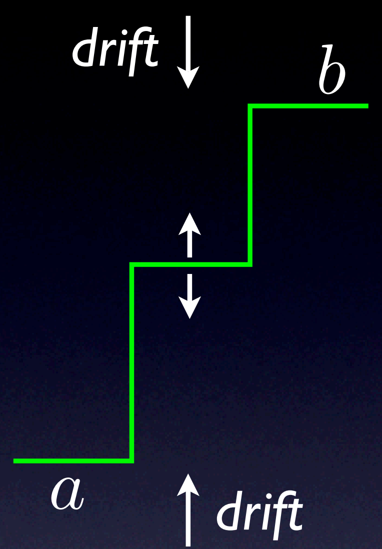
To simulate one move:

- perform $O(t^* n \log n)$ single-site updates, where $t^* =$ mixing time <u>within</u> column
- <u>censor</u> updates in even (odd) columns

[Peres/Winkler]: Censoring can only <u>slow down</u> dynamics

Hence single-site mixing time is $\tilde{O}(n^3 t^*)$

Mixing time within column, t^*



Mixing time $t^* = \tilde{O}((b-a)^2) \leq \tilde{O}(\mathrm{grad}^2)$

where

$$\boxed{\mathrm{grad} = \max_i |\eta(i) - \eta(i-1)|}$$

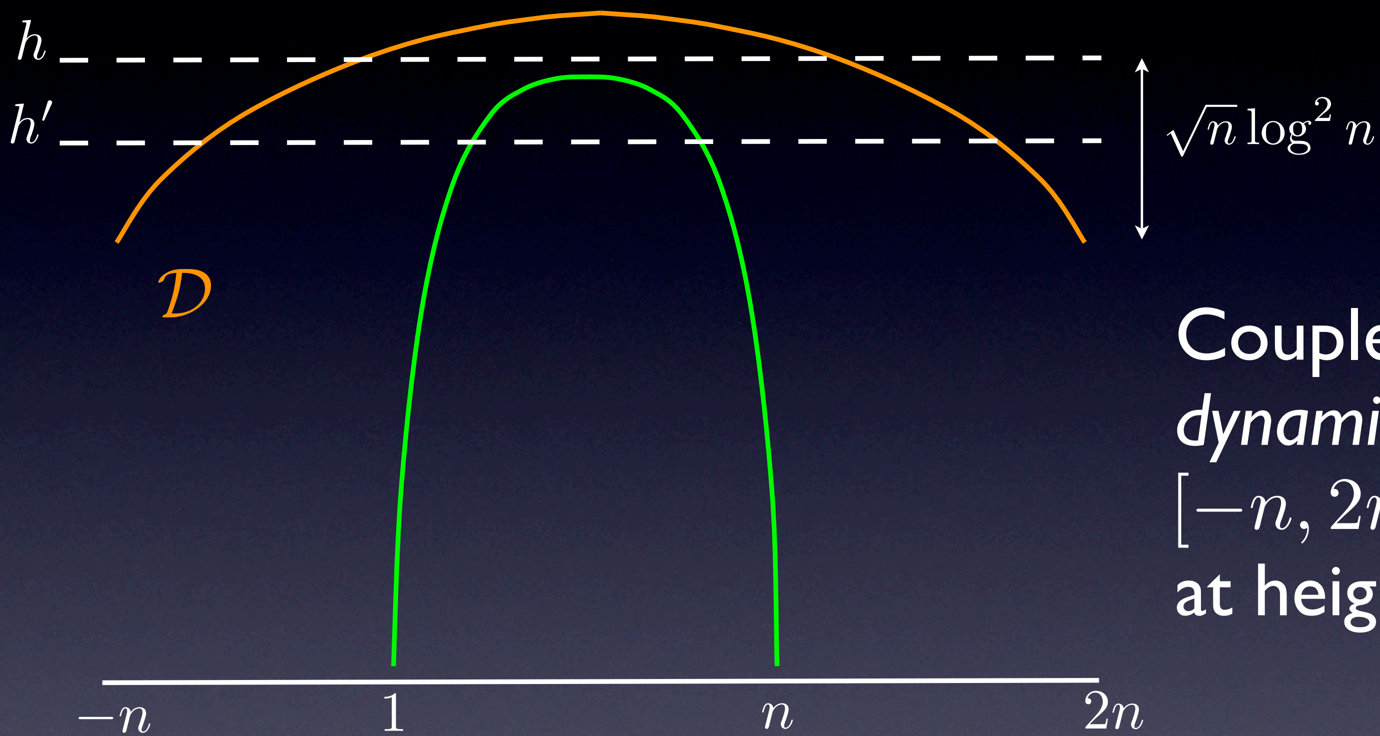
So we need to keep grad small

<u>Problems</u> :

- $\mathrm{grad} = \Theta(n)$ at boundaries !
- Proving <u>non-equilibrium</u> properties for MCs is hard !

Bounding dynamics

Goal: bring contour down from height h to $h' = h - \sqrt{n}$



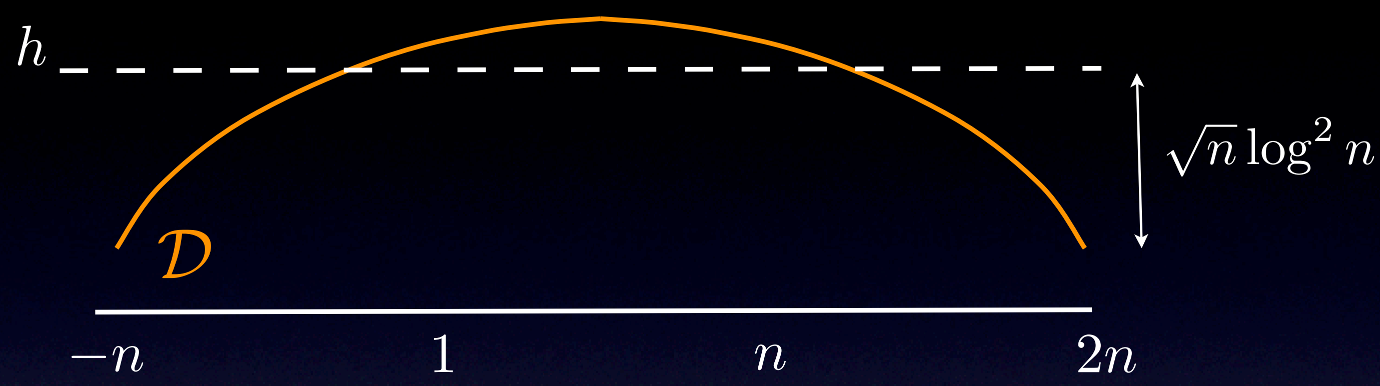
Couple with *bounding dynamics* $\mathcal{D}$ on $[-n, 2n]$ with b.c.'s at height h–$\sqrt{n}\log^2 n$

In equilibrium $\mathcal{D}$ is below h'

Claim: $\mathcal{D}$ has $\mathrm{grad} \leq \mathrm{polylog}(n)$ w.h.p.

Hence: Time to reach small height ($\sim \sqrt{n}$) is $\tilde{O}(n^{3.5})$ ✓

Gradient of bounding dynamics



Event $B = \{\mathrm{grad} > \log^{4.5} n\}$

Easy to see: <u>in equilibrium</u> $\pi(B) \le e^{-c\log^{4.5} n}$

Start in equilibrium <u>conditional on</u> $\overbrace{\text{being above } h \text{ on } [1,n]}^{H}$

Note that $\pi(H) \ge e^{-c\log^4 n}$

Thus at all times t, $\Pr_t(B) \le \pi(B)/\pi(H) \le e^{-c\log^{4.5} n}$ ✓

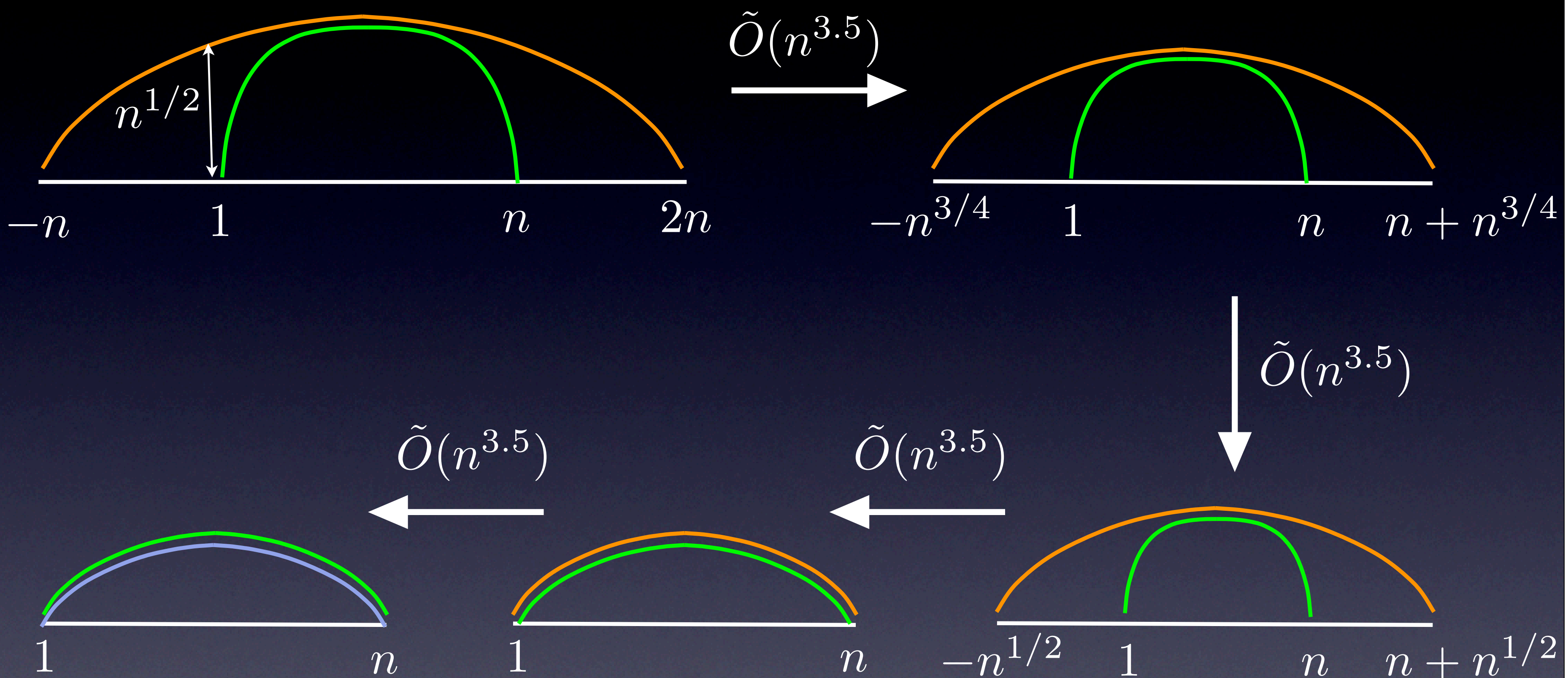
Gradient of bounding dynamics (cont.)

$B = \{\text{grad} > \log^{4.5} n\}$ **(bad event)**

$H = \{\text{above } h \text{ on } [1\text{,n}]\}$ **(conditioning event)**

$$\begin{aligned}
\Pr_t(B) &= \textstyle\sum_{\eta \in B} \sum_{\eta_0} \Pr(\eta_0) \Pr(\eta_t = \eta \mid \eta_0) \\
&= \textstyle\sum_{\eta \in B} \sum_{\eta_0} \frac{\pi(\eta_0)}{\pi(H)} \Pr(\eta_t = \eta \mid \eta_0) \\
&= \textstyle\frac{1}{\pi(H)} \sum_{\eta \in B} \pi(\eta) \\
&= \textstyle\frac{\pi(B)}{\pi(H)}
\end{aligned}$$

Final smoothing



So overall mixing time is $\tilde{O}(n^{3.5})$ ✓

Note: Smoothing phase can be improved to $\tilde{O}(n^3)$

Extensions?

- Match the lower bound of $\Omega(n^3)$
- Adapt to lozenge tilings [Luby/Randall/S., Wilson]
- Extend to low-temperature Ising model (see very recent developments by [Martinelli/Toninelli])
- Censoring + Geometry $\longrightarrow$ analysis of other (monotone) models